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The unsteady motion of a small sphere in a viscous liquid

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The drag experienced by a sphere moving with velocity dependent on a single time scale t_0 in an unbounded viscous liquid is considered under the assumption that the Reynolds number is small. It is shown that, unless t_0 is sufficiently large, an asymptotic expansion in the Reynolds number becomes invalid for large times. Moreover, when the expansion is valid for large times, the drag can differ considerably from that predicted by the unsteady Stokes equations.

1. Introduction

A knowledge of the drag experienced by a small sphere moving with variable velocity in a viscous liquid is important in the study of aerosol motion. In particular, in order to find the relaxation time taken for a sphere subjected to a constant force to attain a constant velocity, it is necessary to know the drag on the sphere for all times. Fuchs (1964) has calculated this relaxation time using the drag predicted by the unsteady Stokes equations in which the non-linear terms in the fluid acceleration are neglected. However, we shall see that, when the sphere's velocity depends on a single time scale t_0, the solution of the unsteady Stokes equations is inaccurate for large times in the sense that the small Reynolds number perturbation of the solution does not tend to the correct steady-state solution found by Proudman & Pearson (1957). More generally, we shall find that any small Reynolds number expansion of the unsteady flow field will only be uniformly valid for all times provided t_0 is sufficiently large. For such values of t_0, we shall show that the drag at all times can differ significantly from that predicted by the unsteady Stokes equations.

We thus consider the motion produced when a sphere of radius a moves without rotation along the z-axis in an unbounded, incompressible fluid of density ρ and kinematic viscosity ν, at rest at infinity. We suppose that at time t the sphere has a velocity $U_0 U(t/t_0)$, where U_0 has the dimensions of velocity. In addition, we shall assume $U \to U_\infty$ as $t \to \infty$ and that $U(0) = 0$, impulsive motion being treated as a limit as $t_0 \to 0$. Then, if we take axes moving with the centre of the sphere and make distances non-dimensional with a, velocities with U_0, pressure with $\rho U_0^2/R$ and time with t_0, the Navier–Stokes equations become

$$\lambda \frac{\partial \mathbf{q}}{\partial t} + R\left[-U(t)\frac{\partial \mathbf{q}}{\partial z} + (\mathbf{q}\,.\,\nabla)\mathbf{q}\right] = -\nabla p + \nabla^2 \mathbf{q}, \quad \nabla\,.\,\mathbf{q} = 0, \tag{1.1}$$

where $\mathbf{q}$ and p denote the velocity and pressure of the fluid, $R = U_0 a/\nu$ and $\lambda = a^2/\nu t_0$. The boundary conditions are

$$\mathbf{q} = U(t)\mathbf{e}_3 \quad \text{on} \quad r = 1, \tag{1.2}$$

$$\mathbf{q} = \mathbf{0} \quad \text{as} \quad r \to \infty, \tag{1.3}$$

and

$$\mathbf{q} = \mathbf{0} \quad \text{at} \quad t = 0, \tag{1.4}$$

where $\mathbf{e}_3$ is a unit vector along the z-axis. By symmetry, $\mathbf{q}$ may be derived from a streamfunction ψ satisfying

$$q_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \quad q_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}.$$

in spherical polar co-ordinates.

We see that the form of an asymptotic solution of (1.1) for small R depends on the relative magnitudes of λ and R. Although, for a given $U(t)$, the flow field is described in terms of $\mathbf{r}$, t, t/λ and R, a result of Meyer (1967) concerning the asymptotic approximation of functions of $\mathbf{r}$, t and R for large t and small R suggests that the asymptotic solution of (1.1) is only valid for all times if λ is sufficiently small. As an example of the way in which the solution could depend on the magnitude of λ, let us first consider the unsteady Stokes equations, obtained by putting $R = 0$ in (1.1). Then the exact solution for the drag (see, for example, Basset (1888)) may be written in non-dimensional form as

$$6\pi\left[U(t) + \left(\frac{\lambda}{\pi}\right)^{\frac{1}{2}}\int_0^t \frac{dU}{d\tau}\frac{d\tau}{(t-\tau)^{\frac{1}{2}}} + \frac{\lambda}{9}\frac{dU}{dt}\right]. \tag{1.5}$$

Assuming further that

$$dU/dt = O(t^{-\gamma}) \tag{1.6}$$

as $t \to \infty$, where $\gamma > \frac{3}{2}$, (1.5) tends to $6\pi U_\infty[1 + (\lambda/\pi t)^{\frac{1}{2}} + \ldots)$ as $t \to \infty$. Now an asymptotic expansion of the drag for large λ, in powers of $\lambda^{-\frac{1}{2}}$, may be found after matching two expansions for the velocity field, using a layer of thickness $\lambda^{-\frac{1}{2}}$ on the sphere. This gives (1.5) in the reverse order, so that the first term is not a uniformly valid approximation for large λ as $t \to \infty$. However, an expansion of the drag for small λ, in powers of $\lambda^{\frac{1}{2}}$, again using matched expansions in regions $r = O(1)$ and $r = O(\lambda^{-\frac{1}{2}})$, yields (1.5) and the first term is now a valid first approximation for all times.

In § 2 we shall consider the dependence of the solution of (1.1) on the magnitude of λ, especially for large times, and in § 3 the particular case $\lambda = O(R^2)$. The matching techniques to be used are similar to those employed by Riley (1966, 1967) in the study of oscillatory viscous flow, where again the flow patterns depend crucially on the relative magnitudes of λ and R. We shall retain assumption (1.6) in order to simplify the analysis, but this is not necessary.

2. The dependence of the solution on the magnitude of λ

(a) λ of order unity or larger

The situation when λ is large is similar to that already described for the Stokes flow. To first order the flow consists of a shear layer of thickness $\lambda^{-\frac{1}{2}}$ on the sphere, surrounded by an outer region of potential flow and the first three terms of the expansion for the drag are (1.5).

When $\lambda = O(1)$, we may proceed by iterating on the unsteady Stokes solution. Putting

$$\psi = \psi_0(\mathbf{r},t) + R\psi_1(\mathbf{r},t) + \ldots \tag{2.1}$$

we find that

$$\left(E^2 - \lambda \frac{\partial}{\partial t}\right) E^2 \psi_0 = 0, \tag{2.2}$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right),$$

with

$$\frac{\partial\psi_0}{\partial\theta} = U(t)\sin\theta\cos\theta, \quad \frac{\partial\psi_0}{\partial r} = U(t)\sin^2\theta \quad \text{on} \quad r = 1,$$

$\psi/r^2 \to 0$ at infinity and $\psi = 0$ at $t = 0$. Thus $\psi_0 = f_0(r,t)\sin^2\theta$, where

$$\bar{f}_0(r,s) = \int_0^\infty f_0(r,t)\, e^{-st} dt = \frac{\bar{U}}{2r\alpha^2}(3 + 3\alpha + \alpha^2) - \frac{3\bar{U}}{2r\alpha^2}(1 + \alpha r)\, e^{-\alpha(r-1)} \tag{2.3}$$

and $\alpha^2 = \lambda s$.

Since the vorticity associated with ψ_0 is exponentially small as $r \to \infty$, matched expansions are not necessary to determine ψ_1, which satisfies

$$\left(E^2 - \lambda \frac{\partial}{\partial t}\right) E^2 \psi_1 = \sin^2\theta\cos\theta F_0(r,t), \tag{2.4}$$

where

$$F_0(r,t) = \left[\frac{2U}{r} - \frac{4f_0}{r^3} + \left(\frac{2f_0}{r^2} - U\right)\frac{\partial}{\partial r}\right]\left[\frac{\partial^2 f_0}{\partial r^2} - \frac{2f_0}{r^2}\right].$$

Thus we try

$$\psi_1 = \sin^2\theta\cos\theta f_1(r,t), \tag{2.5}$$

where

$$\left(\frac{d^2}{dr^2} - \frac{6}{r^2} - \alpha^2\right)\left(\frac{d^2\bar{f}_1}{dr^2} - \frac{6\bar{f}_1}{r^2}\right) = \bar{F}_0(r,s), \tag{2.6}$$

with $\bar{f}_1 = d\bar{f}_1/dr = 0$ on $r = 1$ and $f_1/r^2 \to 0$ as $r \to \infty$. In fact, once $\psi_1/\sin^2\theta\cos\theta$ is known to be a function of r, the associated drag can be shown to vanish by symmetry. We shall, however, determine f_1 approximately in order to justify (2.5), since the results will be useful subsequently.

Although the complicated form of f_0 makes the full solution of (2.6) rather involved, we can find the form of f_1 for large t by taking α small in the expression for $\bar{f}_0$. Since $\bar{f}_0$ does not possess a useful uniformly valid expansion as $\alpha \to 0$ for all r, we consider the cases $r = O(1)$ and $r = O(t^{\frac{1}{2}})$ separately.

When $r = O(1)$, $f_0 = (U_\infty/4r)(3r^2 - 1) + O(t^{-\frac{1}{2}})$ and so

$$\bar{F}_0 = 9\lambda U_\infty^2/4\alpha^2\left(-\frac{2}{r^2} + \frac{3}{r^3} - \frac{1}{4r^5}\right) + O\left(\frac{1}{\alpha}\right),$$

which shows that F_0 tends to its steady flow value as $t \to \infty$. Exactly as in the steady-flow case, if $\bar{f}_1 = O(1/\alpha^2)$ as $\alpha \to 0$, the particular integral of (2.6) for $r = O(1)$ which satisfies the boundary conditions on $r = 1$ grows like

$$-3\lambda U_\infty^2 r^2/16\alpha^2$$

as $r \to \infty$. Also any complementary function which satisfies the boundary conditions grows at least as fast as r^3.

Next, if $r = O(t^{\frac{1}{2}})$, then

$$\bar{f}_0 = \frac{3\lambda U_\infty}{2\alpha^3}\left[\frac{1}{\alpha r} - \left(1+\frac{1}{\alpha r}\right)\exp\{-\alpha r\}\right] + O\left(\frac{1}{\alpha^3 r}\right), \tag{2.7}$$

and so

$$f_0 = \frac{3U_\infty}{2\lambda r}\left\{t - r\left(\frac{\lambda t}{\pi}\right)^{\frac{1}{2}}\exp\{-\lambda r^2/4t\} - \left(t - \frac{\lambda r^2}{2}\right)\operatorname{erfc}\left[\frac{r}{2}\left(\frac{\lambda}{t}\right)^{\frac{1}{2}}\right]\right\} + O\left(\frac{t^{\frac{1}{2}}}{r}\right). \tag{2.8}$$

Thus, after some algebra,

$$F_0 = -\frac{3U_\infty^2}{2}\left[\frac{3}{r^2}\operatorname{erfc}\left[\frac{r}{2}\left(\frac{\lambda}{t}\right)^{\frac{1}{2}}\right] + \left(\frac{\lambda}{\pi t}\right)^{\frac{1}{2}}\left(\frac{3}{r}+\frac{\lambda r}{2t}\right)\exp\{-\lambda r^2/4t\}\right] + O\left(\frac{t^{\frac{1}{2}}}{r^4}\right) \tag{2.9}$$

and so

$$\bar{F}_0 = -\frac{3\lambda U_\infty^2}{2}\left(1+\frac{3}{\tilde{r}}+\frac{3}{\tilde{r}^2}\right)\exp\{-\tilde{r}\} + O(\alpha), \tag{2.10}$$

where $\tilde{r} = \alpha r = O(1)$. Now the solution of

$$\left(\frac{d^2}{d\tilde{r}^2} - \frac{6}{\tilde{r}^2} - 1\right)\left(\frac{d^2\bar{f}_1}{d\tilde{r}^2} - \frac{6\bar{f}_1}{\tilde{r}^2}\right) = -\frac{3\lambda U_\infty^2}{2\alpha^4}\left(1+\frac{3}{\tilde{r}}+\frac{3}{\tilde{r}^2}\right)\exp\{-\tilde{r}\}, \tag{2.11}$$

which approaches zero as $\tilde{r} \to \infty$ is

$$\bar{f}_1 = \frac{A_1}{\tilde{r}^2} + A_2\exp\{-\tilde{r}\}\left(1+\frac{3}{\tilde{r}}+\frac{3}{\tilde{r}^2}\right) + \frac{3\lambda U_\infty^2}{4\alpha^4}\exp\{-\tilde{r}\}(1+\tilde{r}), \tag{2.12}$$

where A_1 and A_2 are arbitrary constants. This solution can only be $O(\tilde{r}^2)$ as $\tilde{r} \to 0$ if $A_1 = -3A_2 = -9\lambda U_\infty^2/2\alpha^4$, in which case it tends to $-3\lambda U_\infty^2 r^2/16\alpha^2$ as $\tilde{r} \to 0$. In view of the behaviour of $\bar{f}_1$ when $r = O(1)$, this is the correct choice for A_1 and A_2 and thus the solution for $\bar{f}_1$ for small α when $r = O(1)$ is

$$\bar{f}_1 = \frac{3\lambda U_\infty^2}{16\alpha^2}\left(-r^2 + \frac{3r}{2} - \frac{1}{2} + \frac{1}{2r} - \frac{1}{2r^2}\right) + O\left(\frac{1}{\alpha}\right). \tag{2.13}$$

This is exactly the Laplace transform of the particular integral of the non-singular perturbation of the steady Stokes solution. As previously mentioned, the symmetry of ψ_1 causes the $O(R)$ contribution to the drag on the sphere to vanish as $t \to \infty$. Consequently λ must be smaller than $O(1)$ as $R \to 0$ for the asymptotic expansion of ψ for small R to be uniformly valid as $t \to \infty$.

(b) $\lambda = O(R)$

For arbitrary small λ the form of the asymptotic expansion for small R is not particularly simple, but the case $\lambda = O(R)$ can be treated fairly easily. Guided by the results for the unsteady Stokes equations, when $\lambda = \kappa_1 R$ we try inner and outer expansions of the form

$$\psi_{\text{inner}} = \psi_0(\mathbf{r},t) + R^{\frac{1}{2}}\psi_1(\mathbf{r},t) + R\psi_2(\mathbf{r},t) + \ldots, \tag{2.14}$$

$$\psi_{\text{outer}} = R^{-\frac{1}{2}}\psi^{(0)}(\mathbf{r}',t) + \psi^{(1)}(\mathbf{r}',t) + \ldots, \tag{2.15}$$

where $\mathbf{r}' = R^{\frac{1}{2}}\mathbf{r}$. The time t then only appears as a parameter in the inner equations and the first few terms in the expansions for ψ can be written down at once. We find

$$\psi_0 = g_0 \sin^2\theta, \tag{2.16}$$

where

$$g_0 = \tfrac{1}{4}U(t)(3r - [1/r]);$$

$$\psi^{(0)} = g^{(0)} \sin^2\theta,$$

where

$$\bar{g}^{(0)} = \frac{3\bar{U}}{2\alpha_1}\left(\frac{1}{\alpha_1 r'} - \left(1 + \frac{1}{\alpha_1 r'}\right)\exp\{-\alpha_1 r'\}\right) \quad (\alpha_1^2 = \kappa_1 s); \tag{2.17}$$

and

$$\psi_1 = g_1 \sin^2\theta, \tag{2.18}$$

where

$$\bar{g}_1 = \frac{\alpha_1 \bar{U}}{2}\left(-r^2 + \frac{3r}{2} - \frac{1}{2r}\right).$$

Thus $\psi^{(1)}$ satisfies

$$\left(E'^2 - \kappa_1 \frac{\partial}{\partial t}\right) E'^2 \psi^{(1)} = \sin^2\theta \cos U(t) \left(\frac{2}{r'} - \frac{d}{dr'}\right)\left(\frac{d^2 g^{(0)}}{dr'^2} - \frac{2g^{(0)}}{r'^2}\right), \tag{2.19}$$

where E'^2 denotes E^2 with r replaced by r', with $\psi^{(1)}/r'^2 \to 0$ as $r' \to \infty$, and, by matching, $\bar{\psi}^{(1)} \to \frac{3}{4}\alpha_1 \bar{U} r' \sin^2\theta$ as $r' \to 0$. The complementary function of (2.19) satisfying both these conditions is the inverse transform of

$$\frac{3\bar{U}}{2}\left[\frac{1}{\alpha_1 r'} - \left(1 + \frac{1}{\alpha_1 r'}\right)\exp\{-\alpha_1 r'\}\right]\sin^2\theta,$$

but the particular integral is only easily written down for large times, under assumption (1.6). Then $\bar{g}^{(0)}$ may be approximated for small α_1 and all r' by replacing $\bar{U}$ by U_∞/s in (2.17), and (2.19) reduces to (2.11) with slight changes in notation. We may therefore conclude that the only particular integral which satisfies the boundary conditions at infinity and is $o(r')$ as $r' \to 0$ tends to

$$-\tfrac{3}{16}U_\infty^2 r'^2 \sin^2\theta \cos\theta + O(t^{-\frac{1}{2}})$$

as $r' \to 0$ for large t and thus dominates the complementary function.

Hence ψ_2, which satisfies

$$E^4\psi_2 = -\frac{3\kappa_1 U'(t)}{2r}\sin^2\theta + \sin^2\theta\cos\theta G_0(r,t), \tag{2.20}$$

where G_0 denotes F_0 with f_0 replaced by g_0, with $\psi_2 = \partial\psi_2/\partial r = 0$ on $r = 1$, must match the inverse transform of

$$-\frac{3\kappa_1 U_\infty^2}{16\alpha_1^2} r^2 \sin^2\theta\cos\theta + O(\alpha_1^{-1})$$

as $r \to \infty$ for small α_1. Thus

$$\bar{\psi}_2 = -\frac{3\bar{U^2}}{16}\left[r^2 - \frac{3r}{2} + \frac{1}{2} - \frac{1}{2r} + \frac{1}{2r^2}\right]\sin^2\theta\cos\theta + O(\alpha_1^{-1}) \tag{2.21}$$

as $\alpha_1 \to 0$.

As in the previous section, the symmetry of $\bar{\psi}_2$ means there is no $O(R)$ term in the drag on the sphere as $t \to \infty$. The two-term expansion for the drag calculated from (2.16) and (2.18) gives the first two terms in (1.5), with λ replaced by $\kappa_1 R$.

To obtain the third term it would have been necessary to calculate $\overline{\psi}_2$ correct to $O(1)$ as $\alpha_1 \to 0$.

In summary, therefore, λ must be smaller than $O(R)$ before a perturbation solution for small R is valid as $t \to \infty$. We shall see in the next section that a uniformly valid expansion is obtained when $\lambda = O(R^2)$.

3. The case $\lambda = O(R^2)$

When $\lambda = \kappa R^2$ we try an inner expansion of the form

$$\psi = \psi_0(\mathbf{r}, t) + R\psi_1(\mathbf{r}, t) + \ldots, \tag{3.1}$$

as this is then the form of the expansion both for steady flow and for unsteady Stokes flow. Again the time only appears as a parameter in the inner solution and $\psi_0 = \frac{1}{4}U(t)(3r - r^{-1})\sin^2\theta$. Also ψ_1, which satisfies the same equation as it would in steady flow, differs from

$$-\frac{3[U(t)]^2}{16}\left(r^2 - \frac{3r}{2} + \frac{1}{2} - \frac{1}{2r} + \frac{1}{2r^2}\right)\sin^2\theta\cos\theta$$

by a complementary function of $E^4\psi_1 = 0$ which vanishes together with its derivative on $r = 1$.

The form of ψ_0 suggests an outer expansion of the form

$$\mathbf{q} = R\mathbf{q}^{(0)}(\mathbf{r}^*, t) + \ldots, \tag{3.2}$$

$$p = R^2 p^{(0)}(\mathbf{r}^*, t) + \ldots, \tag{3.3}$$

$$\psi = \frac{1}{R}\psi^{(0)}(\mathbf{r}^*, t) + \ldots, \tag{3.4}$$

where $\mathbf{r}^* = R\mathbf{r}$. Then

$$\kappa\frac{\partial \mathbf{q}^{(0)}}{\partial t} - U(t)\frac{\partial \mathbf{q}^{(0)}}{\partial z^*} = -\nabla^* p^{(0)} + \nabla^{*2}\mathbf{q}^{(0)}, \quad \nabla^* . \mathbf{q}^{(0)} = 0, \tag{3.5}$$

where $\mathbf{q}^{(0)}$ vanishes as $r^* \to \infty$ and matches the first-order inner solution as $r^* \to 0$. We adopt Childress's (1964) device of assuming that this inner boundary condition is satisfied if $\mathbf{q}^{(0)}$ and $p^{(0)}$ satisfy

$$\kappa\frac{\partial \mathbf{q}^{(0)}}{\partial t} - U(t)\frac{\partial \mathbf{q}^{(0)}}{\partial z^*} = -\nabla^* p^{(0)} + \nabla^{*2}\mathbf{q}^{(0)} + 6\pi U(t)\,\delta(\mathbf{r}^*)\,\mathbf{e}_3, \quad \nabla^* . \mathbf{q}^{(0)} = 0, \tag{3.6}$$

everywhere.

The Fourier transforms of $\mathbf{q}^{(0)}$ and $p^{(0)}$ are defined by

$$\mathbf{Q}^{(0)} = \iiint_{-\infty}^{\infty} \mathbf{q}^{(0)} \exp\{-i\mathbf{k}.\mathbf{r}^*\} d\mathbf{r}^*, \quad P^{(0)} = \iiint_{-\infty}^{\infty} p^{(0)} \exp\{-i\mathbf{k}.\mathbf{r}^*\} d\mathbf{r}^*,$$

where $\mathbf{k} = (k_1, k_2, k_3)$. Thus, if $k = |\mathbf{k}|$,

$$\kappa\frac{d\mathbf{Q}^{(0)}}{dt} - ik_3 U(t)\mathbf{Q}^{(0)} = -iP^{(0)}\mathbf{k} - k^2\mathbf{Q}^{(0)} + 6\pi U(t)\,e_3, \quad \mathbf{k}.\mathbf{Q}^{(0)} = 0. \tag{3.7}$$

Hence $$P^{(0)} = -6\pi i k_3 U(t)/k^2$$
and so, using (1.4),

$$\mathbf{Q}^{(0)} = -\frac{6\pi}{\kappa}\exp\left\{-\frac{k^2 t}{\kappa}+ik_3 Z(t)\right\}\int_0^t U(\tau)\exp\left\{\frac{k^2\tau}{\kappa}-ik_3 Z(\tau)\right\}d\tau\left(\frac{k_3\mathbf{k}}{k^2}-\mathbf{e}_3\right), \tag{3.8}$$

where $\kappa Z'(t) = U(t)$. Moreover, provided U approaches U_∞ algebraically as $t\to\infty$, it can be shown that $\mathbf{Q}^{(0)}$ tends to

$$-\frac{6\pi U_\infty}{k^2 - ik_3 U_\infty}\left(\frac{k_3\mathbf{k}}{k^2}-\mathbf{e}_3\right),$$

as $t\to\infty$ for $k \neq 0$. Since $\mathbf{Q}^{(0)}$ is bounded as $t\to\infty$ when $k = 0$, it follows that $\mathbf{q}^{(0)}$, as given by

$$\mathbf{q}^{(0)} = -\frac{3}{4\kappa\pi^2}\int_0^t U(\tau)\,d\tau\iiint_{-\infty}^{\infty}\exp\left\{-\frac{k^2(t-\tau)}{\kappa}+ik_3(Z(t)-Z(\tau))+i\mathbf{k}\cdot\mathbf{r}^*\right\}d\mathbf{k} \times\left(\frac{k_3\mathbf{k}}{k^2}-\mathbf{e}_3\right) \tag{3.9}$$

tends to its expected steady value as $t\to\infty$, namely the velocity associated with a streamfunction $\psi^{(0)} = \frac{3}{2}(1+\cos\theta)(1-\exp\{-\frac{1}{2}U_\infty[r^*-z^*]\})$. It is shown in appendix A that the streamfunction $\psi^{(0)}$ is then given by

$$\psi^{(0)} = \frac{3r^{*2}\sin^2\theta}{2(\kappa\pi)^{\frac{1}{2}}}\int_0^t\frac{U(\tau)}{r_1^{*2}}\left[\int_0^1\exp\left\{-\frac{\kappa r_1^{*2}\beta^2}{4(t-\tau)}\right\}d\beta-\exp\left\{-\frac{\kappa r_1^{*2}}{4(t-\tau)}\right\}\right]\frac{d\tau}{(t-\tau)^{\frac{1}{2}}}, \tag{3.10}$$

where $r_1^{*2} = r^{*2}+2z^*(Z(t)-Z(\tau))+(Z(t)-Z(\tau))^2$ and β is a dummy variable of integration.

In order to carry out the matching to determine ψ_1, $\psi^{(0)}$ must be expanded in powers of R, having first put $\mathbf{r}^* = R\mathbf{r}$. In appendix B it is shown that

$$\psi^{(0)} = \frac{3U(t)\,r\sin^2\theta}{4}R+\left\{-\tfrac{3}{16}(U(t))^2r^2\sin^2\theta\cos\theta + \frac{r^2\sin^2\theta}{2}\left[-\left(\frac{\kappa}{\pi}\right)^{\frac{1}{2}}\int_0^t\frac{dU}{d\tau}\frac{d\tau}{(t-\tau)^{\frac{1}{2}}} + 3\left(\frac{\kappa}{\pi}\right)^{\frac{1}{2}}\int_0^1 d\beta\int_0^{Z(t)}\left(h_0(Y)+\frac{\kappa(\beta^2-1)\,Y^2}{4T}\right)\frac{dY}{T^{\frac{1}{2}}Y^2}\right]\right\}R^2+O(R^3), \tag{3.11}$$

where

$$T = t-\tau,\quad Y = Z(t)-Z(\tau)\quad\text{and}\quad h_0(Y) = \exp\{-\kappa\beta^2Y^2/4T\}-\exp\{-\kappa Y^2/4T\}.$$

Thus matching requires that the complementary function occurring in ψ_1 should be of the form $\frac{1}{2}H(t)\,(r^2-\frac{3}{2}r+[1/2r])\sin^2\theta$, where $H(t)$ denotes the term in square brackets in (3.11).

The next term in the asymptotic expansion for ψ can be calculated at once by using Proudman & Pearson's results for the steady case. The equation for the third term in the inner expansion for ψ is modified only by the presence of a term $\kappa\partial/\partial t(E^2\psi_0)$. Now the particular integral corresponding to this term is proportional to $r^3 \sin^2\theta dU/dt$. Hence the structure of the third term in the inner expansion for ψ is unaltered. Moreover there still cannot be terms of $O(\log R)$ in the outer expansion. If there were, the corresponding pressure and velocity would satisfy (3.5) everywhere since the force on the sphere contains no $R\log R$ terms. Any such solution of (3.5) which is zero at infinity and at $t = 0$ vanishes

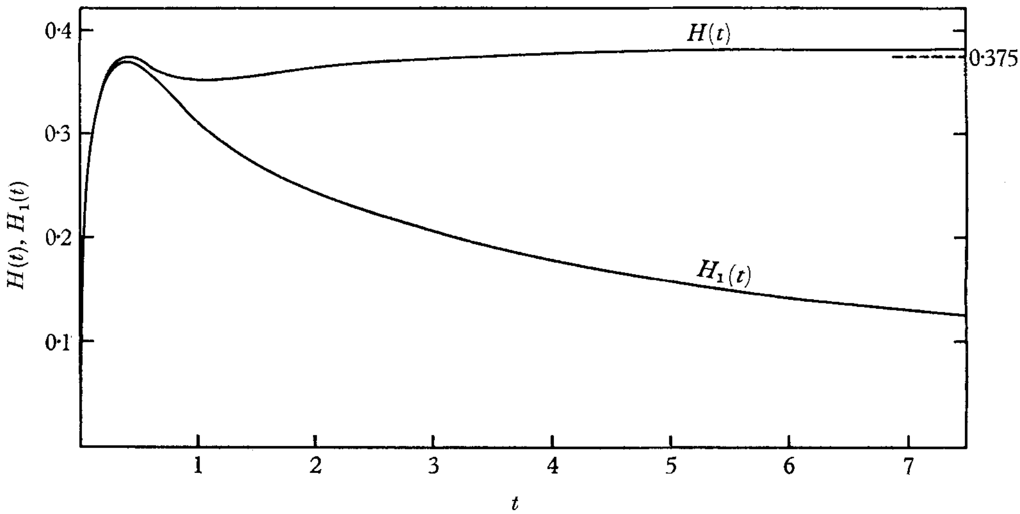

FIGURE 1. The functions $H(t)$ and $H_1(t)$ when $U = 1-1/(1+t)^2$, $\kappa = \frac{1}{4}$.

identically. Therefore we may still infer, as in (3.59) of Proudman & Pearson's paper, that the third term in the inner expansion for ψ is

$$-\tfrac{9}{40}[U(t)]^3 R^2 \log R(\tfrac{1}{2}r^2 - \tfrac{3}{4}r + [1/4r])\sin^2\theta$$

and hence that the non-dimensional drag is

$$6\pi[U(t) - RH(t) + \tfrac{9}{40}[U(t)]^3 R^2 \log R + \ldots]. \tag{3.12}$$

In view of the comments made after (3.8), this expression does tend to the correct steady-state value when $U \to U_\infty$ algebraically. The first term in $H(t)$, denoted by $H_1(t)$, then tends to zero, while the second approaches $-\frac{3}{8}U_\infty^2$. This is demonstrated in figure 1, where $H(t)$ and $H_1(t)$ are compared for the case $U(t) = 1-1/(1+t)^2$, $\kappa = \frac{1}{4}$.

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Appendix A

Let us first consider $u^{(0)}$, the velocity component in the x-direction. From (3.9)

$$u^{(0)} = -\frac{3}{4\kappa\pi^2}\int_0^t U(\tau)\left\{\iiint_{-\infty}^{\infty} \exp\{-w^2k^2+i(k_1x^*+k_2y^*+k_3\zeta)\}\frac{k_1k_3}{k^2}\,d\mathbf{k}\right\}d\tau, \tag{A 1}$$

where $w^2 = (t-\tau)/\kappa$ and $\zeta = z^*+Z(t)-Z(\tau)$. After rotating the axes to a system $\mathbf{k}'$ in which (x^*, y^*, ζ) lies on the k'_3 axis, the integral in curly brackets can be written as

$$\begin{aligned}
&-2\pi\frac{\partial^2}{\partial x^*\partial\zeta}\int_0^\infty\int_0^\pi \exp\{-w^2k'^2+ik'r_1^*\cos\phi\}\sin\phi\,d\phi\,dk' \\
&= -4\pi\frac{\partial^2}{\partial x^*\partial\zeta}\int_0^\infty \frac{\sin k'r_1^*}{k'r_1^*}\exp\{-w^2k'^2\}dk' \\
&= -\frac{2\pi^{\frac{3}{2}}}{w}\frac{\partial^2}{\partial x^*\partial\zeta}\int_0^1 \exp\{-r_1^{*2}\beta^2/4w^2\}d\beta \\
&= \frac{\pi^{\frac{3}{2}}x^*\zeta}{w^3}\left[\left(\frac{1}{r_1^{*2}}+\frac{6w^2}{r_1^{*4}}\right)\exp\{-r_1^{*2}/4w^2\}-\frac{6w^2}{r_1^{*4}}\int_0^1\exp\{-r_1^{*2}\beta^2/4w^2\}d\beta\right].
\end{aligned} \tag{A 2}$$

The y and z velocity components may be treated similarly.

We may prove by direct differentiation that, if a streamfunction

$$\sin^2\theta\int_0^t N(r^*,\tau)\,d\tau \quad \text{gives a velocity} \quad \int_0^t \mathbf{M}(r^*,z^*,\tau)\,d\tau,$$

then the streamfunction

$$r^{*2}\sin^2\theta\int_0^t\frac{N(r_1^*,\tau)\,d\tau}{r_1^{*2}} \quad \text{gives a velocity} \quad \int_0^t \mathbf{M}(r_1^*,\zeta,\tau)\,d\tau.$$

Now the x velocity component associated with the streamfunction

$$\frac{3\sin^2\theta}{2(\kappa\pi)^{\frac{1}{2}}}\int_0^t U(\tau)\left[\int_0^1\exp\{-r^{*2}\beta^2/4w^2\}d\beta-\exp\{-r^{*2}/4w^2\}\right]\frac{d\tau}{(t-\tau)^{\frac{1}{2}}} \tag{A 3}$$

is

$$-\frac{3}{4\kappa\pi^2}\int_0^t U(\tau)\left[\frac{\pi^{\frac{3}{2}}x^*z^*}{w^3}\left\{\left(\frac{1}{r^{*2}}+\frac{6w^2}{r^{*4}}\right)\exp\{-r^{*2}/4w^2\}-\frac{6w^2}{r^{*4}}\int_0^1\exp\{-r^{*2}\beta^2/4w^2\}d\beta\right\}\right]d\tau. \tag{A 4}$$

Hence, from (A 2), (3.10) does indeed give the velocity (3.9).

Appendix B

Let us assume that $Z(0) = 0$ and that $U(t)$ is bounded and continuous for $t \geqslant 0$. If we make the substitutions indicated after (3.11), so that

$$T = \kappa Y/U(t)+O(Y^2) \quad \text{as} \quad Y\to 0,$$

and also put $Q(Y) = Y^2 + 2Rr\cos\theta Y + R^2r^2$, then

$$\psi^{(0)} = \frac{3R^2r^2\sin^2\theta}{2}\left(\frac{\kappa}{\pi}\right)^{\frac{1}{2}}\int_0^{Z(t)}\int_0^1 \frac{h(Y)\,d\beta\,dY}{T^{\frac{1}{2}}Q(Y)}, \tag{B 1}$$

where $h(Y) = \exp\{-\kappa\beta^2Q(Y)/4T\} - \exp\{-\kappa Q(Y)/4T\}$.

When $Y > 0$,

$$\frac{h(Y)}{Q(Y)} = \frac{1}{Y^2}\sum_{n=0}^{\infty} h_n(Y)R^n,$$

where $h_n(Y) = O(Y^{-n+1})$ as $Y \to 0$, and

$$h_0(Y) = \exp\{-\kappa\beta^2Y^2/4T\} - \exp\{-\kappa Y^2/4T\}.$$

Thus we first consider

$$I_1 = \int_\delta^{Z(t)} \frac{h(Y)\,dY}{T^{\frac{1}{2}}Q(Y)}, \tag{B 2}$$

where δ is an arbitrary positive number. Now

$$\int_\delta^{Z(t)}\left(h_0(Y) + \frac{\kappa(\beta^2-1)Y^2}{4T}\right)\frac{dY}{T^{\frac{1}{2}}Y^2} = \int_0^{Z(t)}\left(h_0(Y) + \frac{\kappa(\beta^2-1)Y^2}{4T}\right)\frac{dY}{T^{\frac{1}{2}}Y^2} + O(\delta^{\frac{1}{2}}) \tag{B 3}$$

for small δ, while

$$\int_\delta^{Z(t)}\frac{dY}{T^{\frac{1}{2}}} = \frac{1}{\kappa}\int_0^t \frac{U(t-\tau)\,dT}{T^{\frac{3}{2}}} = \frac{2}{\delta^{\frac{1}{2}}}\left[\frac{U(t)}{\kappa}\right]^{\frac{3}{2}} - \frac{2}{\kappa}\int_0^t \frac{dU}{d\tau}\frac{d\tau}{(t-\tau)^{\frac{1}{2}}} \tag{B 4}$$

since $U(0) = 0$. Thus

$$\int_\delta^{Z(t)}\frac{h_0(Y)\,dY}{T^{\frac{1}{2}}Y^2} = \frac{a_0}{\delta^{\frac{1}{2}}} + b_0 + O(\delta^{\frac{1}{2}}) \tag{B 5}$$

for small δ, where $a_0 = \{[U(t)]^{\frac{3}{2}}/2\kappa^{\frac{1}{2}}\}(1-\beta^2)$ and

$$b_0 = \int_0^{Z(t)}\left(h_0(Y) + \frac{\kappa(\beta^2-1)Y^2}{4T}\right)\frac{dY}{T^{\frac{1}{2}}Y^2} + \frac{\beta^2-1}{2}\int_0^t \frac{dU}{d\tau}\frac{d\tau}{(t-\tau)^{\frac{1}{2}}}.$$

Similarly,

$$\int_\delta^{Z(t)}\frac{h_n(Y)\,dY}{T^{\frac{1}{2}}Y^2} = \frac{a_n}{\delta^{n+\frac{1}{2}}} + O(\delta^{-n+\frac{1}{2}})$$

and so, as $R \to 0$,

$$I_1 = \frac{a_0}{\delta^{\frac{1}{2}}} + b_0 + O(\delta^{\frac{1}{2}}, R). \tag{B 6}$$

We next consider

$$I_2 = \int_0^\delta \frac{h(Y)\,dY}{T^{\frac{1}{2}}Q(Y)}, \tag{B 7}$$

and put $\mu = \frac{1}{4}\beta^2U(t)$ and $\mu' = \frac{1}{4}U(t)$. Throughout the range of integration

$$\frac{h(Y)}{T^{\frac{1}{2}}} = \left[\frac{U(t)}{\kappa Y}\right]^{\frac{1}{2}}[\exp\{-\mu Q(Y)/Y\} - \exp\{-\mu' Q(Y)/Y\}](1 + O(Y)).$$

Exactly as for I_1,

$$\int_\delta^\infty \frac{\exp\{-\mu Q(Y)/Y\} - \exp\{-\mu' Q(Y)/Y\}}{Y^{\frac{1}{2}}Q(Y)}\,dY = \frac{a_0'}{\delta^{\frac{1}{2}}} + b_0' + O(\delta^{\frac{1}{2}}, R) \tag{B 8}$$

as $R \to 0$, where $a_0' = 2(\mu' - \mu)$ and

$$b_0' = \int_0^\infty [\exp\{-\mu Y\} - \exp\{-\mu' Y\} + (\mu - \mu') Y] \frac{dY}{Y^{\frac{5}{2}}}$$
$$= \frac{4\pi^{\frac{1}{2}}}{3} (\mu^{\frac{3}{2}} - \mu'^{\frac{3}{2}}).$$

Next,

$$\frac{\partial}{\partial \mu} \int_0^\infty \frac{\exp\{-\mu Q(Y)/Y\} dY}{Y^{\frac{1}{2}} Q(Y)} = \exp\{-2\mu Rr \cos\theta\} \int_0^\infty \exp\{-\mu(Y + R^2r^2/Y)\} \frac{dY}{Y^{\frac{3}{2}}}$$
$$= -\left(\frac{\pi}{\mu}\right)^{\frac{1}{2}} \frac{\exp\{-2\mu Rr(1+\cos\theta)\}}{Rr},$$

and so

$$\int_0^\infty \frac{\exp\{-\mu Q(Y)/Y\} - \exp\{-\mu' Q(Y)/Y\} dY}{Y^{\frac{1}{2}} Q(Y)}$$
$$= \frac{2\pi^{\frac{1}{2}}}{Rr} (\mu'^{\frac{1}{2}} - \mu^{\frac{1}{2}}) + \frac{4\pi^{\frac{1}{2}}}{3} (1 + \cos\theta)(\mu^{\frac{3}{2}} - \mu'^{\frac{3}{2}}) + O(R) \quad \text{as} \quad R \to 0. \quad \text{(B 9)}$$

Thus, since

$$\int_0^\delta \frac{\exp\{-\mu Q(Y)/Y\} - \exp\{-\mu' Q(Y)/Y\}}{Q(Y)} Y^{\frac{1}{2}} dY = O(\delta^{\frac{1}{2}}) \quad \text{as} \quad R \to 0,$$

$$I_2 = \left[\frac{U(t)}{\kappa}\right]^{\frac{1}{2}} \left\{\frac{[\pi U(t)]^{\frac{1}{2}}}{Rr}(1-\beta) + \frac{\pi^{\frac{1}{2}}[U(t)]^{\frac{3}{2}}}{6} \cos\theta(\beta^3 - 1) - \frac{U(t)(1-\beta^2)}{2\delta^{\frac{1}{2}}} + O(R, \delta^{\frac{1}{2}})\right\}. \quad \text{(B 10)}$$

If we finally let $\delta \to 0$

$$I_1 + I_2 = \left(\frac{\pi}{\kappa}\right)^{\frac{1}{2}} \frac{U(t)(1-\beta)}{Rr} + \left(\frac{\pi}{\kappa}\right)^{\frac{1}{2}} \frac{[U(t)]^2}{6} \cos\theta(\beta^3 - 1) + b_0 + O(R), \quad \text{(B 11)}$$

and so, for small R,

$$\psi^{(0)} = \frac{3R^2r^2\sin^2\theta}{2}\left[\frac{U(t)}{2Rr} - \frac{[U(t)]^2\cos\theta}{8}\right.$$
$$\left. + \left(\frac{\kappa}{\pi}\right)^{\frac{1}{2}} \left\{-\frac{1}{3}\int_0^t \frac{dU}{d\tau}\frac{d\tau}{(t-\tau)^{\frac{1}{2}}} + \int_0^1 d\beta \int_0^{Z(t)} \left(h_0(Y) + \frac{\kappa(\beta^2-1)Y^2}{4T}\right)\frac{dY}{T^{\frac{1}{2}}Y^2}\right\} + O(R)\right], \quad \text{(B 12)}$$

which is (3.11).

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